Large clusters in supercritical percolation

P. S. Grinchuk

A. V. Luikov Heat and Mass Transfer Institute, National Academy of Sciences of Belarus, 15 P. Brovka Street, Minsk 220072, Belarus

(Received 11 October 2001; revised manuscript received 26 April 2002; published 25 July 2002)

The statistical behavior of the size of large finite clusters in supercritical percolation on a finite lattice is investigated (below the critical dimension of the space $d_c=6$). For this purpose, an approximate system of ordinary differential equations for a number of finite clusters is obtained. The correlation between the critical exponents ζ that determine the cluster decay law ($\ln n_s \sim -s^{\zeta}$) and the surface of clusters is shown. It is found that for clusters without self-intersections having a maximal surface $\zeta=1$. For clusters with a small number of self-intersections $\zeta=1-\eta$. Here η is a function depending on the ratio of the surface area of a cluster to its size, which tends to zero, when the surface tends to a maximum. For compact clusters with a minimum or near-minimum surface area, the first correction to the cluster decay law above percolation threshold ($\ln n_s \sim -s^{(d-1)/d}$) has been found on the basis of the drop model and the derived system of equations. The predictions are tested numerically on two- and three-dimensional lattices by Monte Carlo simulations. The results of the work allow one to conclude that above the percolation threshold majority of large clusters are compact and that the cluster surface is the main factor affecting its behavior in supercritical percolation.

DOI: 10.1103/PhysRevE.66.016124 PACS number(s): 64.60.Ak, 02.50.−r

I. INTRODUCTION

Percolation is a standard model for structurally disordered systems. Among numerous areas of its application are polymer gelation, epidemic diseases of garden trees, flows in porous media, hopping conduction in semiconductors, etc. [1,2]. Recently this theory has also been applied to describing social and economic phenomena [3,4].

Among the problems of the percolation theory, the so-called cluster size distribution is of major importance. Many properties of percolative systems can be found by means of this distribution. Typical examples are magnetic susceptibility and the contribution of an external magnetic field to heat capacity of spin glasses [5], magnetization density for diluted ferromagnets [6], susceptibility of a liquid-porous solid system [7], optical absorption edge for diamondlike carbon [8], geometry of the localized wave function in the integer quantum Hall effect [9], etc.

Since the cluster numbers $n_s(p)$ are known exactly for $s \leqslant 10$ [10], this problem may be thought as solved for p far from p_c, because in this case there is a small quantity of large clusters. The study of this problem in the neighborhood of the percolation threshold is of the greatest interest. In one of the recent works [11] this problem was considered in detail below the percolation threshold. Above the percolation threshold, the so-called infinite cluster (IC) is of primary importance for the system behavior. However, in supercritical percolation large finite clusters still exist. Unfortunately, a phenomenological approach dominates up to date in the study of these clusters. Therefore, the development of theoretical and numerical methods for investigating such clusters is an urgent problem.

In this paper, we find an approximate system of ordinary differential equations for a number of finite clusters and apply it to investigate the behavior of large finite clusters above the percolation threshold for $|p-p_c| \ll 1$, i.e., for supercritical percolation. The proposed approach allows us to obtain a refined law of cluster size distribution in supercritical percolation on the basis of more rigorous considerations and also to trace the role of the cluster surface in this law.

II. A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS FOR A NUMBER OF FINITE CLUSTERS

The site problem on a periodic lattice of N sites embedded in the space of subcritical dimension $d<d_c=6$ is considered.

We deal with an ensemble of percolation systems that includes a set of systems (lattices) with all possible values of a site occupation probability p from 0 to 1. A change in p and a related variation of the system properties is described in terms of a transition from a subensemble of the systems, where a site occupation probability is p (subensemble $\{p\}$) to a subensemble wherein this probability is larger by an infinitesimal quantity dp (subensemble $\{p+dp\}$). We note that all the values related to both finite clusters and the infinite one are considered as average with respect to the corresponding subensemble.

Further, we will use the fact that with increase in a site occupation probability, the cluster cannot break into smaller ones (the occupied site cannot become nonoccupied). An important consequence is that with increasing p (random addition of occupied sites to the lattice) larger-size clusters can be formed only from clusters of smaller sizes by their aggregation. Using this fact, after simple calculations we obtain the conditional probability for the transition of an arbitrary *nonoccupied* site to an occupied one,

$$P_1=\frac{dp}{1-p}. \tag{1}$$

Taking into account only the processes of the first order in dp, we have two different ways for the new cluster formation. An arbitrary s cluster (cluster of s sites) is formed either as a result of association of two smaller-size clusters such as $s_1+s_2+1=s$ (the first way) or from a $(s-1)$ cluster when

one vacant site belonging to the perimeter of this cluster becomes occupied (the second way).

Let us consider the first way in detail. We deal with two subensembles of percolation systems: $\{p\}$ and $\{p+dp\}$. The s cluster is formed on an arbitrary site of the system if the following set of conditions is fulfilled: the given site is nonoccupied [probability $(1-p)$]; one of the neighboring site has been ''wetted'' with a s_1 cluster [probability $s_1 n_{s_1}(p)$, where $n_{s_1}(p)$ is the number of s_1 clusters per site], another neighboring site has been ''wetted'' with a s_2 cluster [probability $s_2 n_{s_2}(p)$], with increasing the fraction of occupied sites (transition from $\{p\}$ to the $\{p+dp\}$ subensemble) the considered site becomes occupied [the conditional probability according to Eq. (1)].

Moreover, two given clusters placed in neighboring sites should not intersect with each other nor with any third cluster. We introduce a combinatorial function $g_{s_1 s_2}(p)$ by means of which we take into account the above circumstances. This function is defined as a number of mutual arrangements of all cluster pairs of sizes s_1 and s_2 at two sites neighboring to an arbitrary nonoccupied site of the considered lattice when the clusters do not intersect with each other nor with any third cluster. The number of this arrangements is averaged over all sites of these clusters and over all sites of the lattice. Then, the following expression can be written for the probability of s-cluster formation at an arbitrary site of the lattice through the first way:

$$P_s^{(1+)} = 2 \sum_{s_1 + s_2 + 1 = s} g_{s_1 s_2}(p) s_1 n_{s_1}(p) s_2 n_{s_2}(p) dp. \quad (2)$$

Here factor 2 appears because wetting of each of these two sites with aforementioned clusters is equivalent.

An important detail should be outlined. As can be seen, the formation of a s cluster through the first way will be possible if not one, but for example, three neighboring sites near the considered site belong to the same s_1 cluster. This and similar situations are taken into account by means of the functions $g_{s_1 s_2}(p)$.

By analogy with Eq. (2), for the probability of the s-cluster formation through the second way, we can write

$$P_s^{(2+)} \approx 2(s-1) n_{s-1}(p)(1-p)^{z-1} dp. \quad (3)$$

Here z is the coordination number of the lattice. We note that in Eq. (3) only the most essential term is retained; the situations in which two and more sites nearest to the nonoccupied site belong to the $(s-1)$ cluster are neglected since their probability is proportional to $[(s-1)n_{s-1}(p)]^m$ where $m = 2,3,\ldots,z$ and $(s-1)n_{s-1}(p) \ll 1$.

A decrease in the number of the s clusters is possible owing to the joining of these clusters with any finite cluster or a single site belonging to the perimeter of the s cluster. This is also possible due to the attachment of the s cluster to the IC. The probabilities of a decrease in the number of the s clusters via three aforementioned ways are, correspondingly, the following:

$$P_s^{(1-)} = 2 s n_s(p) \sum_{s_3=1}^{\infty} g_{s s_3}(p) s_3 n_{s_3}(p) dp, \quad (4)$$

$$P_s^{(2-)} \approx 2 s n_s(p)(1-p)^{z-1} dp, \quad (5)$$

$$P_s^{(3-)} = 2 s n_s(p) g(s,p) P(p) dp. \quad (6)$$

Here $P(p)$ is the density of IC or the part of the sites belonging to an IC and $g(s,p)$ is the combinatorial function for a finite and an infinite clusters.

As the next step, the following balance relationship can be written:

$$N n_s(p+dp) - N n_s(p) = N[P_s^{(1+)} + P_s^{(2+)} - P_s^{(1-)} - P_s^{(2-)} - P_s^{(3-)}]. \quad (7)$$

After simple mathematical transformations in the limit $dp \to 0$ we find the system of ordinary differential equations for the number of finite clusters,

$$\frac{dn_s}{dp} \approx 2 \sum_{s_1 + s_2 + 1 = s} [g_{s_1 s_2}(p) s_1 n_{s_1}(p) s_2 n_{s_2}(p)] + 2(1-p)^{z-1}(s-1) n_{s-1}(p) - 2 s n_s \left\{ \sum_{s_3=1}^{\infty} [g_{s s_3}(p) s_3 n_{s_3}(p)] + (1-p)^{z-1} + g(s,p) P(p) \right\}. \quad (8)$$

The developed approach can be applied to an infinite cluster. This will permit us to obtain some information about function $g(s,p)$. We emphasize that with increasing the fraction of the occupied sites, an increase in the fraction of sites belonging to IC and not the formation of new ICs occurs. With account for the first-order processes in dp, there are two different ways of increasing the IC density: (i) joining of an arbitrary finite cluster to IC and (ii) joining of one of the sites along the IC perimeter. When a s cluster joins IC, the IC density increases by $(s+1)/N$. We find, by analogy with Eq. (8), the following equation for IC:

$$\frac{dP}{dp} \approx 2P(p) \left[\sum_{\{s\}} g(s,p)(s+1) s n_s(p) + (1-p)^{z-1} \right]. \quad (9)$$

The correlation length tends to infinity at the percolation threshold. This means that the macroscopic parameters of the system in the vicinity of the percolation threshold become independent of any spatial characteristics. In this case, it is important that such small-scale properties of the system as the structure of finite clusters have no substantial effect on the behavior of the system, in particular, on IC. Therefore, we suppose that in the vicinity of the percolation threshold the IC structure is important, first of all, for the function $g(s,p)$ and this function has only a weak dependence on the

properties of finite clusters. This means that $g(s,p)$ can be factorized near the percolation threshold,

$$g(s,p) \approx g_1(s) g_2(p), \tag{10}$$

and $g_1(s)$ depends only slightly on s. Here $g_1(s)$ corresponds to the layout of the finite cluster and $g_2(p)$ to the IC layout.

As $\lim_{p \to p_c + 0} g(s,p) = 0$, we suppose that the following power law is valid:

$$g_2(p) \propto (p - p_c)^{\lambda}. \tag{11}$$

Then, taking into account Eqs. (10) and (11), we conclude that in the vicinity of the percolation threshold ($|p - p_c| \ll 1$) on the right-hand side of Eq. (9) the term $\Sigma_{\{s\}} s^2 n_s(p) \propto (p - p_c)^{-\gamma}$ dominates, where γ is the mean cluster size exponent. Then, for the IC density from Eq. (9) we have

$$\frac{dP}{dp} \propto P(p)(p - p_c)^{\lambda - \gamma}. \tag{12}$$

This equation has three qualitatively different solutions corresponding to the cases $\lambda < \gamma - 1$, $\lambda = \gamma - 1$, and $\lambda > \gamma - 1$. The correct power-law-like solution [$P(p) \propto (p - p_c)^{\beta}$] is obtained only with $\lambda = \gamma - 1$. For $g_2(p)$ in this case, we find

$$g_2(p) \propto (p - p_c)^{\gamma - 1}. \tag{13}$$

Next, consider the behavior of finite large clusters above the percolation threshold for $|p - p_c| \ll 1$ with the aid of the system of equations (8) and relations (10) and (13).

III. STATEMENT OF THE PROBLEM

Let us consider a situation in the vicinity of the percolation threshold above the latter, i.e., the case of supercritical percolation. As the first approximation, we suppose that the change in the number of s clusters in supercritical percolation is determined only by the ''interaction'' of s clusters with the infinite cluster. According to Eq. (8) it means that for determining the cluster size distribution, the following equation will be solved:

$$\frac{dn_s}{dp} \approx -2 s n_s(p) g(s,p) P(p). \tag{14}$$

It should be noted that a solution is sought in the limit of large s values. The boundary condition for Eq. (14) should be set in a percolation threshold. It is known that n_s in the percolation threshold behaves as follows [12,13]:

$$n_s|_{p = p_c} = C s^{-\tau}, \tag{15}$$

where τ is the Fisher exponent [1,14] and C is the constant independent of s. It should be noted that the power law (15) holds for clusters of a linear extent $R_S \ll \xi$, where ξ is the correlation length. Using Eq. (15) as boundary condition for large clusters is justified because near the percolation threshold ξ is also very large. For the density of IC near the threshold, the well-known scaling relation can be used

$$P(p) \approx C_p (p - p_c)^{\beta}. \tag{16}$$

The main problem consists in the determination of the form of the function $g(s,p)$. This will be made below.

IV. COMBINATORIAL FUNCTION AND THE CLUSTER SIZE DISTRIBUTION IN SUPERCRITICAL PERCOLATION

A. Basic relationship for $g_1(s)$

Turning to the definition of function $g(s,p)$, we can write the following relationship:

$$g(s,p) = \frac{\overline{\alpha_1^{(s)}} + \overline{\alpha_2^{(s)}} + \cdots + \overline{\alpha_s^{(s)}}}{s} \approx g_2(p) \frac{\alpha_1^{(s)} + \alpha_2^{(s)} + \cdots + \alpha_s^{(s)}}{s}. \tag{17}$$

Here $\overline{\alpha_m^{(s)}}$ is the number of possible allocations of the s cluster near a site belonging to the IC perimeter, which is averaged over all the IC perimeter sites provided that the s cluster does not intersect with IC nor with any third cluster. In calculating $\overline{\alpha_m^{(s)}}$, it is necessary to displace the s cluster over the lattice so that its mth site ($m = 1, \ldots, s$) will be a nearest neighbor to a site belonging to the IC perimeter.

Since the dependency of $g(s,p)$ on s and p in the considered case separates, further we can deal only with the $\alpha_m^{(s)}$ values.

For the $(s+1)$ cluster, it is possible to write

$$g_1(s+1) = \frac{\alpha_1^{(s+1)} + \alpha_2^{(s+1)} + \cdots + \alpha_{s+1}^{(s+1)}}{s+1}. \tag{18}$$

It is obvious that the following relationship should be satisfied:

$$\alpha_m^{(s+1)} = \alpha_m^{(s)} - \theta_m^{(s+1)}, \tag{19}$$

where $m = 1,2, \ldots, s$ and $\theta_m^{(s+1)}$ is the decrease in the average number of possible allocations of the s cluster in the neighborhood of IC when s-cluster size is increased by 1. Relation (19) reflects the fact that certain combinations of the mutual allocations of the finite and infinite clusters, which were possible earlier, become prohibited because the site added to the s-cluster will superimpose on the sites already occupied by other clusters.

Substituting Eq. (19) into Eq. (18), we obtain the basic relationship for $g_1(s)$:

$$g_1(s+1) = \frac{\alpha_1^{(s)} - \theta_1^{(s+1)} + \cdots + \alpha_s^{(s)} - \theta_s^{(s+1)} + \alpha_{s+1}^{(s+1)}}{s+1} \frac{s}{s} = g_1(s) \frac{s}{s+1} + \frac{\alpha_{s+1}^{(s+1)}}{s+1} - \frac{\theta}{s+1}. \tag{20}$$

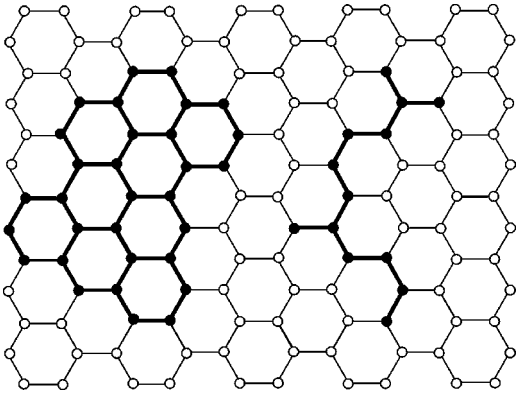

FIG. 1. Examples of compact (left) and noncompact (right) clusters for the honeycomb lattice.

Here $\theta=(\Sigma_{m=1}^{s}\theta_m^{(s)})$ is the total decrease in the number of possible allocations of the s cluster in the neighborhood of IC with increasing the size of the former by 1.

To estimate the order of magnitude of θ, simple numerical simulation has been carried out. A two-dimensional (2D) square lattice is filled with occupied sites in a random way with a prescribed probability $p(|p-p_c|\ll 1)$. An infinite (percolation) cluster is found on the lattice and is labeled. A sufficiently large finite cluster of a given size is generated on another (unoccupied) lattice. After that, each site of this cluster is placed successively onto vacant sites adjacent to the perimeter sites of IC (rotations of a finite cluster were also considered). Thus, all sites of the IC perimeter have been traced. The number of combinations when the finite cluster completely lies on empty sites of the lattice has been calculated. Further, one site is randomly added to this cluster, and all the operations are repeated once again with the same IC. This allows us to estimate the values of both $\alpha_m^{(s+1)}$ and $\theta_m^{(s+1)}$. From the results of simulation it follows that (i) $|\alpha_m^{(s+1)}|\gg|\theta_m^{(s+1)}|$ and (ii) for $s\gg 1$ the value of θ practically does not depend on s. These results will be used below.

We consider two qualitatively different cases: clusters with a small number of self-intersections (noncompact) and compact clusters, i.e., the clusters with a minimal possible, or close to a minimal, surface at a given cluster size. The examples of compact and noncompact clusters for the case of a honeycomb lattice are presented in Fig. 1.

B. Clusters without self-intersections

First of all, we consider a simpler case of clusters without self-intersections. Here, each site of a cluster makes approximately identical contribution to the function $g_1(s+1)$. It means that

$$\alpha_1^{(s+1)}\approx\alpha_2^{(s+1)}\approx\cdots\approx\alpha_s^{(s+1)}\approx\alpha_{s+1}^{(s+1)}. \tag{21}$$

From Eqs. (18) and (21) it follows that for noncompact clusters the relationship $\alpha_{s+1}^{(s+1)}\approx g_1(s+1)$ is valid. Substituting this relationship into Eq. (20) and performing simple mathematical transformations, we obtain the following equation:

$$g_1(s+1)-g_1(s)\approx-\frac{\theta}{s}. \tag{22}$$

Expanding the function $g_1(s+1)$ into the Taylor series in the neighborhood of s $(s\gg 1)$, we derive the ordinary differential equation:

$$\frac{dg_1(s)}{ds}\approx-\frac{\theta}{s}. \tag{23}$$

In this case, the boundary condition for function $g_1(s)$ must be written on the ''right boundary,'' i.e., for $s\to\infty$. For compact clusters, the boundary condition takes the form

$$g_1|_{s=\infty}=0. \tag{24}$$

It means that a very large compact cluster cannot be placed adjacent to IC. In the case of noncompact clusters, the latter assertion is controversial. Therefore, the boundary condition for such clusters is written as follows:

$$g_1|_{s=s_0}=C_0. \tag{25}$$

Formula (25) can be interpreted as a boundary condition for finite clusters of a maximal size, i.e., if $s>s_0$, then $g_1(s)=0$. This means that if $s>s_0$, such a cluster on the given lattice is an infinite one. It should be noted that generally s_0 depends on the lattice size. Here, exact numerical values for s_0 and C_0 are of no significance. It is important only that these values are positive.

Returning to Eq. (23), we write its solution with boundary condition (25):

$$g_1(s)\approx C_0+\ln\left(\frac{s_0}{s}\right)^{\theta}. \tag{26}$$

Substituting Eqs. (10), (13), (16), and (26) into Eq. (14) and taking into account the boundary condition (15), we obtain the following expression for $n_s(p)$:

$$n_s(p)\approx C_1 s^{-\tau}\exp\left(-\frac{s}{s_\xi}\right)\{1+O(s^{-\theta})\}, \tag{27}$$

where C_1 is constant and

$$s_\xi\sim(p-p_c)^{-(\gamma+\beta)}. \tag{28}$$

The quantity s_ξ in Eq. (27) is known as the ''crossover size'' [1,11]. Since large clusters are fractal objects, the crossover size and the correlation length ξ are related as $s_\xi\propto\xi^D$, where D is the fractal dimension of the IC for $p=p_c$. Since the correlation length near the percolation threshold behaves as $\xi\propto(p-p_c)^{-\nu}$, the following expression for D from Eq. (28) is obtained: $D=(\gamma+\beta)/\nu$. This is a well-known result [1,15]. We note that the case of clusters without self-intersections formally corresponds to the Bethe lattices. Then it follows from Eq. (27) that we have obtained a correct critical exponent for these lattices $\zeta=1$ [1].

C. Cluster with a small number of self-intersections

Now let us consider the case of noncompact clusters with a small number of self-intersections. Here, the sites that are enclosed with occupied sites on all sides (i.e., internal sites of a cluster) will not contribute to the function $g_1(s)$. We define the quantity Ω as the ratio of the number of cluster sites s_b, which have one or more vacant neighbor, to the total number of sites in the cluster: $\Omega = s_b/s$. Along with a cluster perimeter, this quantity is a measure of the cluster surface. As s is large, we suppose that the sites that contribute to $g_1(s)$ are characterized by approximately equal contributions.[1] Then, taking into account Eq. (18), we have

$$\alpha_{s+1}^{(s+1)} \approx \frac{g_1(s+1)}{\Omega}. \tag{29}$$

Substituting Eq. (29) into (20) and carrying out simple mathematical transformations, we obtain the following equation:

$$\frac{dg_1}{ds} \approx \frac{\eta}{s-\eta} g_1(s) - \frac{\theta}{s-\eta}, \tag{30}$$

where $\eta = (1/\Omega) - 1$. For compact clusters $\eta > 1$ while for noncompact ones with small number of self-intersections $\eta \to 0$ and $\eta = 0$ for clusters without self-intersections.

The boundary condition for Eq. (30) is similar to condition (25) for Eq. (23): $g_1|_{s=s_0} = C_0'$. Solution of Eq. (30) for fixed η can be written as follows:

$$g_1(s) \approx \left(\frac{s_0-\eta}{s-\eta}\right)^{\eta} \left\{ C_0' + \frac{\theta}{\eta}\left[\left(\frac{s-\eta}{s_0-\eta}\right)^{\eta} - 1\right]\right\}. \tag{31}$$

Taking into account Eqs. (10), (13), (14)–(16), and (31) along with the relationship $\lim_{\eta\to 0}(1-x^{-\eta})/\eta = \ln x$, following expression for the case of small number of self-intersections can be written:

$$n_s(p) \approx C_1' s^{-\tau} \exp\left(-\frac{s^{1-\eta}}{s_\xi}\right)\{1 + O(s^{-\theta})\}, \tag{32}$$

where C_1' is constant and s_ξ is the same as in Eq. (28).

D. Compact clusters

Let us consider the case of compact clusters using the drop model. Then $\Omega \approx A(s+1)^{-1/d}$ and $\eta \approx 1/A(s+1)^{1/d} - 1$, where d is the spatial dimension and $A = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of a sphere of a unit radius in d-dimensional space. Then Eq. (30) takes the following form:

$$\frac{dg_1}{ds} \approx \frac{\frac{1}{A}(s+1)^{1/d} - 1}{s+1-\frac{1}{A}(s+1)^{1/d}} g_1(s) - \frac{\theta}{s+1-\frac{1}{A}(s+1)^{1/d}}. \tag{33}$$

Equation (33) is solved approximately with the boundary condition (24) for subcritical dimensionalities of the space $d = 2$ to 5. Generalizing the obtained solutions the following expression for $g_1(s)$ can be written:

$$g_1(s) \approx A\theta s^{-1/d} + \frac{(d-1)A^2\theta}{d} s^{-2/d} + O(s^{-3/d}). \tag{34}$$

Then from Eqs. (10), (13), (14)–(16), and (34) for $n_s(p)$ we have

$$n_s(p) \approx C_1'' s^{-\tau} \exp\left(-\frac{s^{(d-1)/d}}{s_\xi} - \frac{A(d-1)}{d}\frac{s^{(d-2)/d}}{s_\xi}\right), \tag{35}$$

where C_1'' is constant and s_ξ is the same as in Eq. (28).

It follows from expressions (26), (32), and (35) that for fixed p and large s the number of clusters with a greater Ω will be smaller in comparison with the number of clusters of the same size but having a smaller Ω. Hence, in supercritical percolation the majority of large finite clusters located on a lattice will be compact, and the cluster size distribution will be determined only by expression (35). It should be noted that Eq. (35) without the second term in the exponential function is a widely used empirical expression for the cluster size distribution in supercritical percolation [1,16]. This result agrees also with the results of numerical experiments [17] and the exact inequalities are proven in Ref. [18].

V. NUMERICAL SIMULATION

To verify the predictions of the preceding section, Monte Carlo simulation for a site problem has been carried out on a periodic 2D square lattices with sizes $N = 100^2$ and 1000^2 for $p = 0.59\,274\,621$ (the percolation threshold [19]), 0.60, 0.62, and 0.64, and on 3D simple cubic lattices with sizes $N = 50^3$ and 100^3 for $p = 0.3\,116\,080$ (the percolation threshold [20]), 0.320 and 0.340. The boundary conditions for all the lattices are free. For each combination of (N,p) from 2×10^5 to 2×10^7 histories were treated. The central processing unit time on a dual-processor Pentium III computer with 800 MHz processors was about 2500 h.

For numerical simulation, the well-known Hoshen-Kopelman cluster-labeling algorithm [21] has been modified. It was supplemented with the possibility of calculating not only the size of any cluster but also the number of sites with one or more vacant neighbors, i.e., of determining the quantity Ω. It turned out that this value for $s \gg 1$ lies in a narrow range (Fig. 2). Thus, for a 2D square lattice $\Omega = 0.82$–0.86 at $p = 0.60$ and, respectively, $\eta = 0.78$–0.84 in Eq. (32). This indicates that the occurrence of large clusters with a branched surface is a rare event for supercritical percolation. This result agrees with works [22], where it has been shown

[1] We may assume that each cluster site makes a contribution to the function $g_1(s)$ and this contribution is proportional to the number of the vacant sites adjacent to the given site. Unfortunately, this additional assumption does not give anything new since additional information about the structure of finite clusters is necessary for using it.

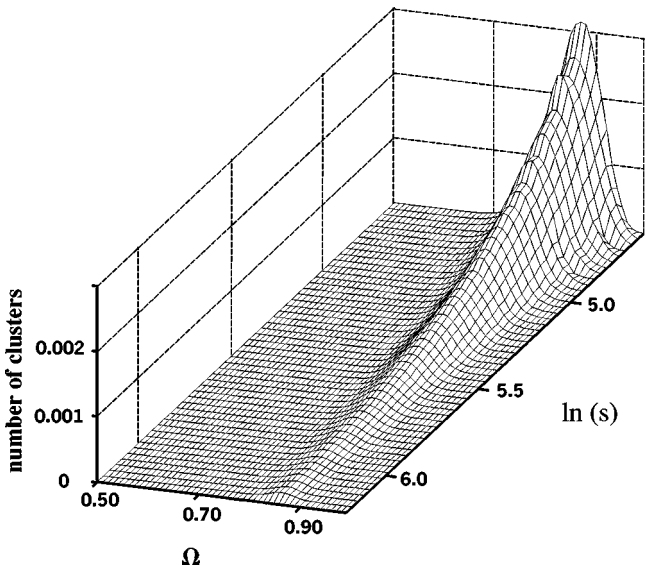

FIG. 2. Number of clusters per site as function of their sizes and values of Ω on a two-dimensional square lattice for $p=0.60$ and $N=100^2$.

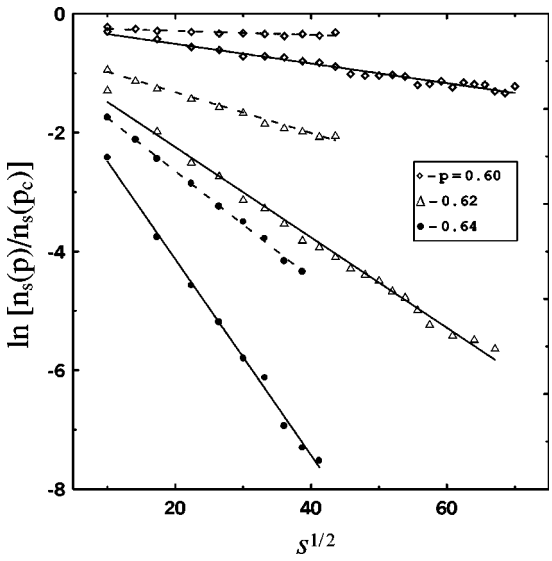

FIG. 4. Verification of the cluster decay law (35) for a 2D square lattice as plot of $\ln[n_s(p)/n_s(p_c)]$ versus $s^{1/2}$. For solid curves $N=1000^2$, for dashed curves $N=100^2$.

that the perimeter distribution function narrows to a δ function if $s\to\infty$. All the aforesaid confirms the conclusions of Sec. IV about the cluster size distribution law.

To verify formula (32) Monte Carlo simulations have been carried out on a 2D square lattice. For this purpose, the quantity Ω is divided into 100 equal parts. For example, the value $\Omega=0.78$ is assigned to all the clusters for which Ω lies in the range 0.78 ± 0.005. The results of simulations are presented in Fig. 3. It can be seen from the Fig. 3 that the behavior predicted by formula (32) is the case.

The fact that all large clusters are characterized by the same Ω for fixed p allowed us to use the standard Hoshen-Kopelman algorithm for verifying Eq. (35). It should be noted that in the two-dimensional case the last term in the exponential function of Eq. (35) is independent of s. Therefore, the cluster size distribution should be linear in the following coordinates: abscissa $s^{1/2}$ and ordinate $\ln[n_s(p)/n_s(p_c)]$. Results of Monte Carlo simulation for square lattices of various sizes are presented in Fig. 4, where it can be seen that the linear dependence in these coordinates really takes place. As is seen from Fig. 4, the straight lines corresponding to the same p but different lattice sizes L differ (approximately twice in logarithmic scale). Estimating the scaling effects for the investigated problem we have found that these effects can introduce an error that is not more than 1–2%. Really, the displacement of a percolation threshold on the lattice of finite size L has the order of $|p_c(L)-p_c(\infty)| \sim 1/L^{\nu}$. In the case of exponential dependence of cluster numbers on a cluster size [Eq. (35)] the correction on scaling (in logarithmic scale) is $\Delta\{\ln[n_s(p)/n_s(p_c)]\}\sim s^{\xi}L^{-D}$. For investigated lattice sizes ($L\sim10^2$–10^3) and cluster sizes ($s\sim10^2$–10^3) the upper estimation for this cor-

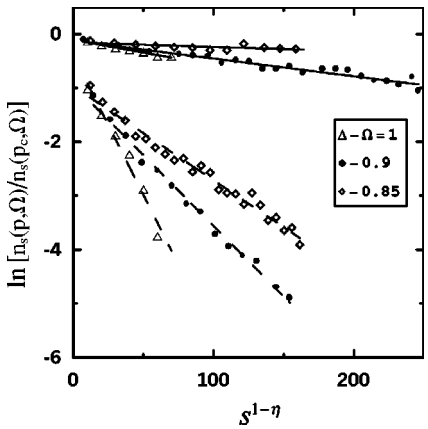

FIG. 3. Verification of the behavior of the cluster sizes predicted by Eq. (32) as plot of $\ln[n_s(p,\Omega)/n_s(p_c,\Omega)]$ versus $s^{1-\eta}$ (where $\eta=(1/\Omega)-1$), for different Ω. For solid curves $p=0.60$, for dashed curves $p=0.64$.

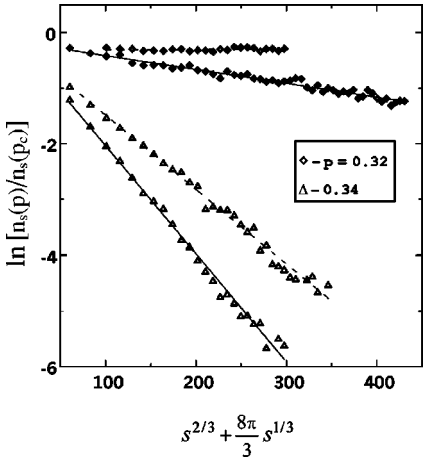

FIG. 5. Verification of the cluster decay law (35) for a simple 3D cubic lattice as a plot of $\ln[n_s(p)/n_s(p_c)]$ versus $s^{2/3}+(8\pi/3)s^{1/3}$. For solid curves $N=100^3$, for dashed curves $N=50^3$.

rection is 1–2%. The discrepancy between the numerical data for various sizes of a lattice is explained by a small absolute value of cluster numbers for large s [16]. Thus, for number of histories $\sim 10^7$ an absolute error of determination of cluster numbers is $\sim 10^{-4}$–10^{-5}. For large s this error is comparable to cluster numbers. For moderately s ($\sim$100–300) the relative errors are 10–20%, and for large s ($\sim$1000) they reach 50–100%. It should be noted that in spite of a large relative error, the linear dependence of average cluster numbers in coordinates mentioned does take place.

In the case of three-dimensional lattices, the second term in Eq. (35) is no longer constant. In the range of the clusters sizes $s \sim 10^3$–10^4 the second term is comparable with the first one in the order of magnitude. To verify relationship (35) for a simple 3D cubic lattice, we also plot the cluster size distribution in special coordinates: the abscissa $s^{2/3} + (8\pi/3)s^{1/3}$ and the ordinate $\ln[n_s(p)/n_s(p_c)]$. As is seen from Fig. 5, there is a linear dependence for the cluster size distribution in the mentioned coordinates at different p and N. It should be noted that here better consistency with the linear dependences is observed than in the case of using $s^{2/3}$ as the abscissa for the aforementioned distribution.

VI. CONCLUSION

In this paper, the statistical behavior of the sizes of large finite clusters for supercritical percolation has been considered. It is demonstrated once again that the cluster surface is of primary importance for the cluster behavior. The suggested approach has allowed us to consider from the general point of view such a characteristic of a percolation system as the cluster size distribution above the percolation threshold. The next logical step in this direction should consist in the replacement of the drop model for compact clusters with a more realistic model. We suppose that further development of this approach will allow better understanding and solving a number of problems of the percolation theory.

ACKNOWLEDGMENTS

This work was supported in part by the Belarusian Republican Fund for Fundamental Research (Grant No. T00-171).

[1] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, 2nd ed. (Taylor & Francis, London, 1995).
[2] M. Sahimi, *Applications of Percolation Theory* (Taylor & Francis, London, 1994).
[3] S. Solomon *et al.*, Physica A **277**, 239 (2000).
[4] J. Goldenberg *et al.*, Physica A **284**, 335 (2000).
[5] A. A. Abrikosov, Pis'ma Zh. Eksp. Teor. Fiz. **27**, 696 (1978).
[6] M. V. Bakri and D. Stauffer, Phys. Rev. B **14**, 4215 (1976).
[7] V. D. Borman, A.M. Grekhov, and V. I. Troyan, Zh. Éksp. Teor. Fiz. **118**, 193 (2000) [Sov. Phys. JETP **91**, 170 (2000)].
[8] M. Pelton *et al.*, J. Appl. Phys. **83**, 1029 (1998).
[9] I. Bratberg, A. Hansen, and E. H. Hauge, Europhys. Lett. **37**, 19 (1997).
[10] M. F. Sykes and M. Glen, J. Phys. A **9**, 87 (1976); M. F. Sykes, D. S. Gaunt, and M. Glen, *ibid.* **9**, 97 (1976); D. Stauffer, Phys. Rev. Lett. **35**, 394 (1975).
[11] M. Z. Bazant, Phys. Rev. E **62**, 1660 (2000).
[12] J. Feder, *Fractals* (Plenum Press, New York, 1988).
[13] D. C. Rapaport, J. Stat. Phys. **66**, 679 (1992).
[14] M. E. Fisher, Physics (Long Island City, N.Y.) **3**, 255 (1967).
[15] A. Kapitulnik, Y. Gefen, and A. Aharony, J. Stat. Phys. **36**, 807 (1984).
[16] *Monte Carlo Methods in Statistical Physics*, edited by K. Binder (Springer-Verlag, Berlin, 1979).
[17] D. Stauffer, Phys. Rev. Lett. **41**, 1333 (1978).
[18] H. Kunz and B. Souillard, Phys. Rev. Lett. **40**, 133 (1978).
[19] M. E. J. Newman and R. M. Ziff, Phys. Rev. Lett. **85**, 4104 (2000).
[20] C. D. Lorenz and R. M. Ziff, J. Phys. A **31**, 8147 (1998); H. G. Ballesteros *et al.*, *ibid.* **32**, 1 (1999).
[21] J. Hoshen and R. Kopleman, Phys. Rev. B **14**, 3438 (1976).
[22] P. L. Leath and G. R. Reich, J. Phys. C **11**, 1155 (1977); **11**, 4017 (1978).